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Research Article

Sectional Category of the Ganea Fibrations and Higher Relative Category

Jean-Paul Doeraene

Département de Mathématiques, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France

Correspondence should be addressed to Jean-Paul Doeraene; doeraene@math.univ-lille1.fr

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We first compute James' sectional category (secat) of the Ganea map g_k of any map ι_X in terms of the sectional category of ι_X: we show that secat g_k is the integer part of secat $\iota_X/(k+1)$. Next we compute the relative category (relcat) of g_k. In order to do this, we introduce the relative category of order k (relcat$_k$) of a map and show that relcat g_k is the integer part of relcat$_k\iota_X/(k+1)$. Then we establish some inequalities linking secat and relcat of any order: we show that secat $\iota_X \leqslant$ relcat$_k\iota_X \leqslant$ secat $\iota_X + k + 1$ and relcat$_k\iota_X \leqslant$ relcat$_{k+1}\iota_X \leqslant$ relcat$_k\iota_X + 1$. We give examples that show that these inequalities may be strict.

1. Introduction

The "Lusternik-Schnirelmann category" cat$X$ of a topological space X is the least integer $n \geqslant 0$ such that X can be covered by $n+1$ open subsets U_i $(0 \leqslant i \leqslant n)$ such that each inclusion $U_i \hookrightarrow X$ is nullhomotopic; that is, the based path-space fibration $PX \to X$ has a partial section on U_i. More generally, the "sectional category" secat p of a fibration $p : E \to X$, originally defined by Schwarz [1], is the least integer $n \geqslant 0$ such that X can be covered by $n+1$ open subsets with a partial section of p on each of these sets. This notion extends to any continuous map $\iota_X : A \to X$ by taking the standard homotopy replacement of ι_X by a fibration $p : E \to X$ and setting secat ι_X = secat p. So cat$X$ = secat$(* \to X)$. Sectional category earned its renown recently as Farber's notion of "topological complexity" [2] of a space A, which measures the difficulty of solving the motion planning problem: the topological complexity of A is the sectional category of the diagonal $\Delta : A \to A \times A$ or equivalently of the (unbased) fibration $\pi : A^I \to A \times A : \alpha \mapsto (\alpha(0), \alpha(1))$.

For a given space X, Ganea [3] defined a sequence of fibrations $p_k : E_k \to X$ for $k \geqslant 0$, starting with $p_0 : PX \to X$. The fundamental property of the sequence is that it gives another criterion for detecting the category: cat$X$ is the least n such that p_n has a section (at least for a sufficiently nice space: normal, well pointed). This construction can be generalized for any map $\iota_X : A \to X$; that is, there is a sequence of maps $g_k(\iota_X) : G_k(\iota_X) \to X$, starting with $g_0(\iota_X) = \iota_X$, and secat(ι_X) is the least n such that $g_n(\iota_X)$ has a homotopy section; see Definition 3. We recover the Ganea construction when $A = *$; in this case we write $g_k(X)$ instead of $g_k(\iota_X)$.

In this paper, we first show that the sectional category of kth Ganea map $g_k(X)$ of X is the integer part of cat$X/(k+1)$. More generally, the sectional category of the Ganea map $g_k(\iota_X)$ associated with any map ι_X is the integer part of secat $\iota_X/(k+1)$.

As we may "think of" the sectional category as the degree of obstruction for a map to have a homotopy section, this shows us how this degree of obstruction decreases when we consider the successive Ganea maps. For instance, for a space X with cat$X = 7$, the successive values of secat$(g_k(X))$ for $0 \leqslant k \leqslant 7$ are

$$7\ \ 3\ \ 2\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0. \tag{1}$$

In [4], we used the same Ganea-type construction to define the "relative category" of a map (relcat for short). As a particular case, the relative category of the diagonal map $\Delta : X \to X \times X$ is the "monoidal topological complexity" of X defined in [5]. It turns out that the relative category can differ from the sectional category by at most one. More precisely, we have

$$\operatorname{secat}\iota_X \leqslant \operatorname{relcat}\iota_X \leqslant \operatorname{secat}\iota_X + 1. \tag{2}$$

This establishes a dichotomy between maps: those for which the sectional category equals the relative category and those for which they differ by 1.

In this paper we introduce the "relative category of order k" (relcat_k) and show that the relative category of kth Ganea map $g_k(\iota_X)$ associated with a map ι_X is the integer part of $\operatorname{relcat}_k\iota_X/(k+1)$. When $\iota_X : * \to X$, we write $\operatorname{relcat}_k\iota_X = \operatorname{cat}_k X$.

Warning. Despite cat_k is sometimes used in the literature for Fox's k-dimensional category, this is *not* the meaning of this notation in this paper.

We link all these invariants together by several inequalities:

$$\begin{gathered}\operatorname{secat}\iota_X \leqslant \operatorname{relcat}_k\iota_X \leqslant \operatorname{secat}\iota_X + k + 1,\\ \operatorname{relcat}_k\iota_X \leqslant \operatorname{relcat}_{k+1}\iota_X \leqslant \operatorname{relcat}_k\iota_X + 1.\end{gathered} \tag{3}$$

Finally, we show that, with some hypothesis on the connectivity of ι_X and the homotopical dimension of the source of $g_k(\iota_X)$, $\operatorname{relcat}_j\iota_X = \operatorname{secat}\iota_X$ for all $j \leqslant k$.

For a given space X (resp.: map ι_X), the set of integers k for which the equality $\operatorname{cat}_{k+1}X = \operatorname{cat}_k X$ (resp., $\operatorname{relcat}_{k+1}\iota_X = \operatorname{relcat}_k\iota_X$) holds is an interesting datum about this space (resp., map). The maximum number of such integers is $\operatorname{cat}X$ (resp., $\operatorname{relcat}\iota_X$). For instance, for $X = K(\mathbb{Q}, 1)$, there is just one such k, which is 0: namely,

$$\begin{gathered}\operatorname{cat}_0 X = \operatorname{cat}_1 X = 2,\\ \operatorname{cat}_k X = k + 1 \quad \text{for } k > 1.\end{gathered} \tag{4}$$

2. Sectional Category of the Ganea Maps

We use the symbol $\simeq$ both to mean that maps are homotopic and to mean that spaces are of the same homotopy type. We denote the integer part of a rational number q by $\lfloor q \rfloor$.

We build all our spaces and maps with "homotopy commutative diagrams," especially "homotopy pullbacks" and "homotopy pushouts," in the spirit of [6].

Recall the following construction.

Definition 1. For any map $\iota_X : A \to X$, the *Ganea construction* of ι_X is the following sequence of homotopy commutative diagrams ($i \geqslant 0$):

$$\begin{tikzcd} & A \arrow[dr, "\alpha_{i+1}"] \arrow[drr, bend left, "\iota_X"] & & \\ F_i \arrow[ur, "\eta_i"] \arrow[dr, "\beta_i"'] & & G_{i+1} \arrow[r, dashed, "g_{i+1}"] & X \\ & G_i \arrow[ur, "\gamma_i"] \arrow[urr, bend right, "g_i"'] & & \end{tikzcd}$$

where the outside square is a homotopy pullback, the inside square is a homotopy pushout, and the map $g_{i+1} = (g_i, \iota_X) : G_{i+1} \to X$ is the whisker map induced by this homotopy pushout. The iteration starts with $g_0 = \iota_X : A \to X$.

In other words, the map g_{i+1} is the join of g_i and ι_X over X; namely, $g_{i+1} \simeq g_i \bowtie_X \iota_X$. When we need to be precise, we denote G_i by $G_i(\iota_X)$ and g_i by $g_i(\iota_X)$. If $A \simeq *$, we also write $G_i(X)$ and $g_i(X)$, respectively.

Notice that, as the outside square is a homotopy pullback, g_i and η_i have a common homotopy fiber, so their connectivity is equal.

For coherence, let $\alpha_0 = \mathrm{id}_A$. For any $i \geqslant 0$, there is a whisker map $\theta_i = (\mathrm{id}_A, \alpha_i) : A \to F_i$ induced by the homotopy pullback. Thus, θ_i is a homotopy section of η_i. Moreover, we have $\gamma_i \circ \alpha_i \simeq \alpha_{i+1}$.

Proposition 2. *For any map $\iota_X : A \to X$, we have*

$$g_j\left(g_i\left(\iota_X\right)\right) \simeq g_{ij+i+j}\left(\iota_X\right). \tag{5}$$

Proof. This is just an application of the "associativity of the join" (see [7, Theorem 4.8], for instance):

$$\begin{aligned} g_j\left(g_i\left(\iota_X\right)\right) &\simeq g_i\left(\iota_X\right) \bowtie_X \cdots \bowtie_X g_i\left(\iota_X\right) \quad (j+1 \text{ times}) \\ &\simeq \left(\iota_X \bowtie_X \cdots \bowtie_X \iota_X\right) \cdots \left(\iota_X \bowtie_X \cdots \bowtie_X \iota_X\right) \\ &\simeq \iota_X \bowtie_X \cdots \bowtie_X \iota_X \\ &\qquad\qquad\qquad ((j+1)(i+1) \text{ times}) \\ &\simeq g_{(j+1)(i+1)-1}\left(\iota_X\right). \end{aligned} \tag{6}$$

□

Definition 3. Let $\iota_X : A \to X$ be any map.

(1) The *sectional category* of ι_X is the least integer n such that the map $g_n : G_n(\iota_X) \to X$ has a homotopy section: that is, there exists a map $\sigma : X \to G_n(\iota_X)$ such that $g_n \circ \sigma \simeq \mathrm{id}_X$.

(2) The *relative category* of ι_X is the least integer n such that the map $g_n : G_n(\iota_X) \to X$ has a homotopy section σ and $\sigma \circ \iota_X \simeq \alpha_n$.

We denote the sectional category by $\operatorname{secat}(\iota_X)$ and the relative category by $\operatorname{relcat}(\iota_X)$. If $A \simeq *$, $\operatorname{secat}(\iota_X) = \operatorname{relcat}(\iota_X)$ and it is denoted simply by $\operatorname{cat}(X)$; this is the "normalized" version of the Lusternik-Schnirelmann category.

A lot about the integers cat and secat is collected in [8]. The integer relcat is introduced in [4] and further studied in [9, 10].

Proposition 4. *For any map $\iota_X : A \to X$, we have*

$$\operatorname{secat} g_k\left(\iota_X\right) = \left\lfloor \frac{\operatorname{secat}\iota_X}{k+1} \right\rfloor. \tag{7}$$

Proof. By definition, $\operatorname{secat} g_k(\iota_X)$ is the least integer n such that $g_n(g_k(\iota_X))$, that is, $g_{kn+k+n}(\iota_X)$, has a homotopy section. Thus, if $\operatorname{secat}\iota_X = m$, n will be such that $kn + k + n \geqslant m$ and $k(n-1) + k + (n-1) < m$: that is, $n \geqslant m/(k+1) - k/(k+1)$ and $n < m/(k+1) + 1/(k+1)$, so $n = \lfloor m/(k+1) \rfloor$. □

3. Higher Relative Category

For any map $\iota_X : A \to X$ and two integers $0 \leqslant k < r$, consider the following homotopy commutative diagram:

$$\begin{tikzcd}
& G_k \arrow[d, "\gamma_k^r"'] \arrow[rr, bend left, "g_k"] & & \\
H_k^{r-k-1} \arrow[ur] \arrow[dr] & G_r \arrow[rr, dashed, "g_r"] & & X \\
& G_{r-k-1} \arrow[u] \arrow[urr, bend right, "g_{r-k-1}"'] & &
\end{tikzcd}$$

where the outside square is a homotopy pullback and the inside square is a homotopy pushout.

Because of the associativity of the join, we also have $\gamma_k^r \simeq \gamma_{r-1} \circ \gamma_{r-2} \circ \cdots \circ \gamma_{k+1} \circ \gamma_k$. For coherence, let $\gamma_k^k = \mathrm{id}_{G_k}$.

Definition 5. Let $\iota_X : A \to X$ be any map. The *relative category of order* k of ι_X is the least integer $n \geqslant k$ such that the map $g_n : G_n(\iota_X) \to X$ has a homotopy section σ and $\sigma \circ g_k \simeq \gamma_k^n$.

We denote this integer by $\mathrm{relcat}_k \iota_X$. In order to avoid the prefix "rel" when $A \simeq *$, we write $\mathrm{cat}_k X = \mathrm{relcat}_k \iota_X$ in this case.

Remark 6. Notice that $\mathrm{relcat}_0 \iota_X = \mathrm{relcat}\, \iota_X$ and that, clearly, $k \leqslant \mathrm{relcat}_k \iota_X \leqslant \mathrm{relcat}_{k+1} \iota_X$ for any k. Also notice that $\mathrm{relcat}_k \iota_X = k$ if and only if $g_k(\iota_X)$ is a homotopy equivalence. In particular, $\mathrm{cat}_k * = k$ for any k.

Following the same reasoning as in Proposition 4, we have the following.

Proposition 7. *For any map $\iota_X : A \to X$, we have*

$$\mathrm{relcat}\, g_k(\iota_X) = \left\lfloor \frac{\mathrm{relcat}_k \iota_X}{k+1} \right\rfloor. \tag{8}$$

Proposition 8. *For any map $\iota_X : A \to X$, any k, we have*

$$\mathrm{secat}\, \iota_X \leqslant \mathrm{relcat}_k \iota_X \leqslant \mathrm{secat}\, \iota_X + k + 1. \tag{9}$$

Proof. Only the second inequality needs a proof. Let $n = \mathrm{secat}\, \iota_X$ and let σ be a homotopy section of g_n. Consider the following homotopy commutative diagram:

$$\begin{tikzcd}
G_k \arrow[r, "\sigma'"] \arrow[dd, "g_k"'] & H_k^n \arrow[rr, "g'"] \arrow[dd] & & G_k \arrow[dl, "\gamma_k^{n+k+1}"'] \arrow[dd, "g_k"] \\
& & G_{n+k+1} \arrow[dr, dashed, "g_{n+k+1}"] & \\
X \arrow[r, "\sigma"'] & G_n \arrow[ur, "\gamma_n^{n+k+1}"] \arrow[rr, "g_n"'] & & X
\end{tikzcd}$$

where $\sigma' = (\sigma \circ g_k, \mathrm{id}_{G_k})$ is the whisker map induced by the right homotopy pullback. We have $g' \circ \sigma' \simeq \mathrm{id}_{G_k}$ and the left square is a homotopy pullback by the Prism lemma (see [7, Lemma 1.3], for instance). The map $\sigma^+ = \gamma_n^{n+k+1} \circ \sigma$ is a homotopy section of g_{n+k+1} and, moreover, $\sigma^+ \circ g_k \simeq \gamma_k^{n+k+1} \circ g' \circ \sigma' \simeq \gamma_k^{n+k+1}$. So $\mathrm{relcat}_k \iota_X \leqslant n + k + 1$. □

Theorem 9. *For any map $\iota_X : A \to X$, any k, we have*

$$k \leqslant \mathrm{relcat}_k \iota_X \leqslant \mathrm{relcat}_{k+1} \iota_X \leqslant \mathrm{relcat}_k \iota_X + 1. \tag{10}$$

Proof. The first two inequalities are our Remark 6; only the third needs a proof. Let $n = \mathrm{relcat}_k \iota_X$ and let σ be a homotopy section of g_n such that $\sigma \circ g_k \simeq \gamma_k^n$. Consider the following homotopy commutative diagram:

$$\begin{tikzcd}
& & F_k \arrow[dll] \arrow[d] \arrow[drr] \arrow[dd, bend left] & & \\
A \arrow[rr, "\bar{\sigma}"] \arrow[dr] \arrow[ddd] & & F_n \arrow[rr, "\eta_n"] \arrow[ddd] & & A \arrow[dl] \arrow[ddl] \arrow[ddd] \\
& G_{k+1} \arrow[ddl, dashed] \arrow[rr, equal, bend right] & G_k \arrow[l] \arrow[r] \arrow[ddll] \arrow[dd] \arrow[ddrr] & G_{k+1} \arrow[d, dashed] & \\
& & & G_{n+1} \arrow[dr, dashed] & \\
X \arrow[rr, "\sigma"'] & & G_n \arrow[ur, "\gamma_n"] \arrow[rr, "g_n"'] & & X
\end{tikzcd}$$

The map $\sigma^+ = \gamma_n \circ \sigma$ is a homotopy section of g_{n+1} and $\sigma^+ \circ g_{k+1} \simeq \gamma_{k+1}^{n+1}$, so $\mathrm{relcat}_{k+1} \iota_X \leqslant n + 1$. □

So $\mathrm{relcat}_k \iota_X$ increases at most by one when k increases by one.

Corollary 10. *For any map $\iota_X : A \to X$, any k, we have*

$$\mathrm{relcat}\, \iota_X \leqslant \mathrm{relcat}_k \iota_X \leqslant \mathrm{relcat}\, \iota_X + k. \tag{11}$$

Remark 11. As a consequence of Theorem 9 and Corollary 10, if $n = \mathrm{relcat}\, \iota_X$, there are at most n integers k for which $\mathrm{relcat}_{k+1} \iota_X = \mathrm{relcat}_k \iota_X$.

Example 12. If ι_X is a homotopy equivalence, then g_k is a homotopy equivalence for all k. So $\mathrm{relcat}_k \iota_X = k$ for all k.

Example 13. Let $A \not\simeq *$ and consider the map $\iota_* : A \to *$. We have $\mathrm{secat}\, \iota_* = 0$ because ι_* has a (unique) section. By Proposition 8, $\mathrm{relcat}_k \iota_* = k$ or $1+k$. Indeed, for any k, the map $\gamma_k^{k+1} : A \bowtie \cdots \bowtie A$ ($k+1$ times) $\to A \bowtie \cdots \bowtie A$ ($k+2$ times) is homotopic to the null map, so $\sigma \circ g_k \simeq \gamma_k^{k+1}$, where $\sigma : * \to G_{k+1}(\iota_*)$. But we cannot have $\mathrm{relcat}_k \iota_* = k$ unless $g_k(\iota_*) : A \bowtie \cdots \bowtie A$ ($k+1$ times) $\to *$ is a homotopy equivalence.

If we choose a space A such that $A \not\simeq *$ but $\Sigma A \simeq *$ (the 2-skeleton of the Poincaré homology 3 spheres, for instance), then $A \bowtie A \simeq \Sigma A \wedge A \simeq *$ and g_k is a homotopy equivalence for all $k > 0$, so $\mathrm{relcat}_0 \iota_* = 1$ and $\mathrm{relcat}_k \iota_* = k$ for all $k > 0$. However, if we chose a simply connected CW-complex A (in order that $A \bowtie \cdots \bowtie A \not\simeq *$), then $\mathrm{relcat}_k \iota_* = k + 1$ for all k.

Example 14. Consider any CW-complex X with $\mathrm{cat} X = 1$ and the map $\iota_X : * \to X$. We have $\mathrm{secat}\, \iota_X = \mathrm{relcat}\, \iota_X = \mathrm{cat} X = 1$.

Let us compute $\mathrm{cat}_1 X = \mathrm{relcat}_1 \iota_X$. Notice that $G_1(X) \simeq \Sigma\Omega X$. By Theorem 9, we know that $1 \leqslant \mathrm{cat}_1 X \leqslant 2$. But we cannot have $\mathrm{cat}_1 X = 1$ because g_1 is not a homotopy equivalence, so $\mathrm{cat}_1 X = 2$. By the way, we can say that $\gamma_1^2 : \Sigma\Omega X \to G_2(X)$ factorizes up to homotopy through $g_1 : \Sigma\Omega X \to X$.

Example 15. More generally, if $\mathrm{relcat}\, \iota_X = 1$, we have $k \leqslant \mathrm{relcat}_k \iota_X \leqslant 1 + k$ for any k by Corollary 10. Thus, $\mathrm{relcat}_k \iota_X = k + 1$ while $g_k(\iota_X)$ is not a homotopy equivalence (and if any n exists such that $g_n(\iota_X)$ is a homotopy equivalence, then $\mathrm{relcat}_k \iota_X = k$ for all $k \geqslant n$).

Suppose we are given any map $\iota_X : A \to X$ with $\mathrm{secat}(\iota_X) \leqslant n$ and any homotopy section $\sigma : X \to G_n$ of $g_n : G_n \to X$. For any $k \leqslant n$, consider the following homotopy pullbacks:

$$\begin{tikzcd}
Q \arrow[r, "\pi"] \arrow[d, "\pi'"'] & G_k \arrow[d, "\theta^n_k"'] \arrow[dr, equal] & \\
G_k \arrow[r, "\overline{\sigma}"'] \arrow[d, "g_k"'] & H^n_k \arrow[r, "\eta^k_n"'] \arrow[d, "\eta^n_k"'] & G_k \arrow[d, "g_k"] \\
X \arrow[r, "\sigma"'] & G_n \arrow[r, "g_n"'] & X
\end{tikzcd}$$

where $\theta^n_k = (\gamma^n_k, \mathrm{id}_{G_k})$ is the whisker map induced by the homotopy pullback H^n_k. Notice that $\eta^k_n \circ \theta^n_k \simeq \mathrm{id}_{G_k}$. By the Prism lemma, we know that the homotopy pullback of σ and η^n_k is indeed G_k and that $\eta^k_n \circ \overline{\sigma} \simeq \mathrm{id}_{G_k}$. Also notice that $\pi \simeq \pi'$ since $\pi \simeq \eta^k_n \circ \theta^n_k \circ \pi \simeq \eta^k_n \circ \overline{\sigma} \circ \pi' \simeq \pi'$.

Proposition 16. *For any map $\iota_X : A \to X$ with $\mathrm{secat}(\iota_X) \leqslant n$ and any homotopy section $\sigma : X \to G_n$ of $g_n : G_n \to X$, with the same definitions and notations as above, the following conditions are equivalent:*

(i) $\sigma \circ g_k \simeq \gamma^n_k$.

(ii) *π has a homotopy section.*

(iii) *π is a homotopy epimorphism.*

(iv) $\theta^n_k \simeq \overline{\sigma}$.

Proof. We have the following sequence of implications:

(i) $\Rightarrow$ (ii): since $\sigma \circ g_k \simeq \gamma^n_k \simeq \eta^n_k \circ \theta^n_k \circ \mathrm{id}_{G_k}$, we have a whisker map $(g_k, \mathrm{id}_{G_k}) : G_k \to Q$ induced by the homotopy pullback Q which is a homotopy section of π.

(ii) $\Rightarrow$ (iii): it is obvious.

(iii) $\Rightarrow$ (iv): we have $\theta^n_k \circ \pi \simeq \overline{\sigma} \circ \pi' \simeq \overline{\sigma} \circ \pi$ since $\pi \simeq \pi'$. Thus, $\theta^n_k \simeq \overline{\sigma}$ since π is a homotopy epimorphism.

(iv) $\Rightarrow$ (i): we have $\sigma \circ g_k \simeq \eta^n_k \circ \overline{\sigma} \simeq \eta^n_k \circ \theta^n_k \simeq \gamma^n_k$.

□

Theorem 17. *Let $\iota_X : A \to X$ be a $(q-1)$ connected map. If for some $k \leqslant \mathrm{secat}\, \iota_X$, G_k has the homotopy type of a CW-complex with dimension strictly less than $(\mathrm{secat}\, \iota_X + 1)q - 1$, then $\mathrm{relcat}_j \iota_X = \mathrm{secat}\, \iota_X$ for all $j \leqslant k$.*

This is an immediate consequence of the following.

Proposition 18. *Let $\iota_X : A \to X$ be a $(q-1)$ connected map with $\mathrm{secat}\, \iota_X \leqslant n$. If for some $k \leqslant n$, G_k has the homotopy type of a CW-complex with dimension strictly less than $(n+1)q-1$, then $\sigma \circ g_k \simeq \gamma^n_k$ for any homotopy section σ of g_n, so $\mathrm{relcat}_k \iota_X \leqslant n$.*

Proof. Recall that, for any $i \geqslant 0$, g_i is the $(i+1)$-fold join of ι_X. Thus, by [11, Theorem 47], we obtain that $g_i : G_i \to X$ is $(i+1)q-1$-connected. As g_i and η^k_i have the same homotopy fiber, which is $(i+1)q-2$-connected, we see that $\eta^k_i : H^i_k \to G_k$ is $(i+1)q-1$-connected, too. By [12, Theorem IV.7.16], this means that, for every CW-complex K with $\dim K < (i+1)q-1$, η^k_i induces a one-to-one correspondence $[K, H^i_k] \to [K, G_k]$. Apply this to $K \simeq G_k$ and $i = n$: since θ^n_k and $\overline{\sigma}$ are both homotopy sections of η^k_n, we obtain $\theta^n_k \simeq \overline{\sigma}$, and Proposition 16 gives the desired result. □

Example 19. Let X be the Eilenberg-Mac Lane space $K(\mathbb{Q}, 1)$. It is known that $\mathrm{cat}(X) = 2$ and that $G_1(X) \simeq \Sigma\Omega X$ has the homotopy type of a wedge of circles (see [8, Example 1.9 and Remark 1.62], for instance). By Theorem 9, we know that $2 \leqslant \mathrm{cat}_1 X \leqslant 3$. Because $\dim G_1(X) = 1 < (\mathrm{cat} X + 1) - 1 = 2$, we have $\sigma \circ g_1 \simeq \gamma_1^2$ for any homotopy section σ of $g_2(X)$ and, thus, $\mathrm{cat}_1 X = 2$. Moreover, g_k is never a homotopy equivalence, so $\mathrm{cat}_k X > k$ for any k; thus, $\mathrm{cat}_k X = k + 1$ for $k \geqslant 1$.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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